The Riemann hypothesis and Hardy's theorem

This project is intended for a student with a firm basis in Complex Analysis and who enjoyed the first part of the undergraduate course Analytic Number Theory relating to the Riemann zeta function.

The number of prime numbers in the interval $[1, x]$, denoted by $\pi(x)$, can be calculated approximately by the prime number theorem, which states that

$$\lim_{x \to +\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1.$$

A very natural question regards the speed with which the limit converges to 1. This was the question that Riemann tried to answer in 1859 by studying the properties of the function $\zeta(s)$. This function is defined for complex numbers s satisfying $\Re(s) > 1$ through

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Firstly he showed that $\zeta(s)$ can also be defined for all complex numbers s with $\Re(s) \leq 1$ (via a different expression) and furthermore that it has infinitely many roots s with real part in the range $0 \leq \Re(s) \leq 1$. His fundamental discovery was that $\zeta(s)$ satisfies a *functional equation*, this is a mechanism that allows to predict the value of ζ at $1-s$ by the one at s. One implication of the functional equation is that the roots of $\zeta(s)$ are symmetric around the line $\Re(s) = \frac{1}{2}$; this means that if s is one of these roots then $1 - s$ is also a root (and vice versa). He went on to observe that the properties of ζ are closely connected to the distribution of prime numbers. It was later proved that the prime number theorem is equivalent to the statement that all of these roots never have real part $\Re(s) = 1$ or $\Re(s) = 0$; this claim was investigated in detail during this year's undergraduate course in *Analytic Number Theory*.

Riemann finished his paper by conjecturing that all of these roots will lie on the symmetry line, i.e. that their real part will always fulfill $\Re(s) = \frac{1}{2}$; this has a vast array of important consequences and therefore his conjecture subsequently became known as the *Riemann Hypothesis*. Despite 150 years of continued research his conjecture has resisted a proof or a disproof. One consequence of his claim is that $\pi(x)$ can be approximated by the elementary function

$$\int_3^x \frac{\mathrm{d}t}{\log t}$$

with an error term which, for every $\epsilon > 0$, is as small as $O_\epsilon(x^{\frac{1}{2}+\epsilon})$. A fairly standard argument shows that this is the best approximation one can wish for and in fact, this error term is equivalent to the Riemann Hypothesis.

The offered project regards studying a modern version of the proof of Hardy's Theorem, who in 1914 proved that *there are infinitely many roots* s *of* $\zeta(s)$ *with real part* $\Re(s) = \frac{1}{2}$. This theorem provides one of the best results towards the validity of the Riemann Hypothesis up to this date. An accessible exposition may be found in [1, Chapter 15] or [2, Section 24.1]. It is based on the functional equation of $\zeta(s)$, certain properties of Dirichlet polynomials and bounds for $|\zeta(\frac{1}{2} + it)|$ when $t \in \mathbb{R}$ and $t \to +\infty$. The prerequisites are a first course in Complex Analysis and the first part of the Analytic Number Theory course.

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References

[1] Iwaniec, H. *Lectures on the Riemann zeta function.* University Lecture Series, 62. *American Mathematical Society, Providence, RI*, 2014.

[2] Iwaniec, H. and Kowalski, E. *Analytic number theory.* American Mathematical Society Colloquium Publications, 53. *American Mathematical Society, Providence, RI*, 2004.